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Analyticity of the Density of States in the Anderson Model on the Bethe Lattice

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Let $H = \frac{1}{2}\Delta + V$ on $l^2(\mathbf{B})$, where $\mathbf{B}$ is the Bethe lattice and $V(x)$, $x \in \mathbf{B}$, are i.i.d.r.v.'s with common probability distribution μ. It is shown that for distributions sufficiently close to the Cauchy distribution, the density of states $\rho(E)$ is analytic in a strip about the real axis.

KEY WORDS: Anderson model; Bethe lattice; random Schrödinger operator; density of states.

1. INTRODUCTION

The Bethe lattice $\mathbf{B}$ is an infinite connected graph with no closed loops and a fixed degree (number of nearest neighbors) at each vertex (site or point). The degree is called the coordination number and the connectivity k is one less the coordination number.

The distance between two sites x and y will be denoted by $|x-y|$ and is equal to the length of the shortest path connecting x and y. The finite volume Λ_l will consist of all sites whose distance from a chosen origin is less than or equal to l. The boundary of Λ_l, $\partial\Lambda_l$, will consist of nearest neighbor pairs $\langle x, y\rangle$ such that $x \in \Lambda_l$ and $y \notin \Lambda_l$.

The Anderson model on the Bethe lattice is given by the random Hamiltonian $H = H_0 + V$ on $l^2(\mathbf{B}) = \{u: \mathbf{B} \to \mathbf{C} \mid \sum_{x\in\mathbf{B}} |u(x)|^2 < \infty\}$, where

$$(H_0 u)(x) = \frac{1}{2} \sum_{y:|x-y|=1} u(y) \tag{1.1}$$

and $V(x)$, $x \in \mathbf{B}$, are independent and identically distributed random variables with common probability distribution μ. The characteristic function of μ will be denoted by h, i.e., $h(t) = \int e^{-itv}\, d\mu(v)$.

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Let χ_l denote the characteristic function of Λ_l. Define the measure dN_l by

$$\int f(\lambda)\, dN_l(\lambda) = \frac{1}{|\Lambda_l|} \operatorname{tr}(f(H)\, \chi_l) \tag{1.2}$$

and the measure dN by

$$\int f(\lambda)\, dN(\lambda) = \mathbf{E}(f(H)(0, 0)) \tag{1.3}$$

for any bounded measurable function f. It is a consequence of the ergodic theorem (see Appendix) that dN_l converges vaguely to dN **P**-a.s. (see, e.g., ref. 1). The integrated density of states $N(E)$ is then defined by

$$N(E) = \int \chi_{(-\infty, E]}(\lambda)\, dN(\lambda) \tag{1.4}$$

and the density of states is given by $\rho(E) \equiv N'(E)$.

For the case $k = 1$ (the one-dimensional case), the integrated density of states is a continuous function,[2–4] being log Holder continuous if the condition $\int \log(1 + |v|)\, d\mu(v) < \infty$ holds.[3] Both Campanino and Klein[5] and March and Sznitman[6] proved in the one-dimensional case that if $h(t)$ is exponentially bounded, then $N(E)$ is analytic in a strip about the real axis. Kunz and Souillard[7] have announced results on the Bethe lattice concerning the analyticity of the density of states for distributions close to the Cauchy distribution, but no proofs have been provided.

For the Cauchy distribution, i.e., $d\mu = (\lambda/\pi)(\lambda^2 + x^2)^{-1}\, dx$, the density of states can be computed explicitly and is analytic for $|\mathrm{Im}\, E| < \lambda$. In this article we study the analyticity of the density of states for distributions sufficiently close to the Cauchy distribution. Our conditions will be stated in terms of h, the characteristic function of μ. We will only be interested in $h(t)$ for $t \geqslant 0$ and we will only consider the right-hand-side derivatives at $t = 0$.

Introducing the Banach spaces

$$S_\alpha = \left\{ f\colon [0, \infty) \to \mathbf{C} \text{ absolute continuous} \,\middle|\, \|f\|_{S_\alpha} = \sum_{k=0}^{1} \|e^{\alpha t} f^{(k)}(t)\|_\infty < \infty \right\}$$

where $\alpha > 0$ and $\|f\|_\infty = \operatorname{ess\,sup}_{t \in [0, \infty)} |f(t)|$, we can now state our result.

Theorem 1.1. Let h_0 be the characteristic function of the Cauchy distribution with parameter λ, i.e., $h_0(t) = e^{-\lambda t}$ for $t \geqslant 0$. Then, for any $\delta > \frac{1}{2}(k+1)$, there exists a neighborhood U around 0 in the space S_δ such that if $h - h_0 \in U$, then $\rho(E)$ is analytic in the strip $|\mathrm{Im}\, E| < \varepsilon$ for some $\varepsilon > 0$.

We will study the density of states by analyzing the Green's function of H. The Green's function corresponding to our Hamiltonian H is given by

$$G(x, y; z) = \langle x \,|(H-z)^{-1}|\, y\rangle \tag{1.5}$$

for $x, y \in \mathbf{B}$ and $\mathrm{Im}\, z > 0$. We will write $G(z) = \mathbf{E}(G(0, 0; z))$.

Since $N(E) = \mathrm{E}\langle 0|\, P(-\infty, E]\, |0\rangle$, where P is the spectral projection of H, we have that $G(z)$ is the Borel transform of $N(E)$ (e.g., refs. 8 and 9), i.e.,

$$G(z) = \int \frac{dN(E)}{E - z}$$

and we have the following:

(i) $G(E + i0) = \lim_{\eta \downarrow 0} G(E + i\eta)$ exists for a.e. $E \in R$.

(ii) Suppose $G(E + i0)$ exists for all E in an open interval I; then $N(E)$ is absolutely continuous and $\rho(E) = (1/\pi)\, \mathrm{Im}\, G(E + i0)$.

Thus, to obtain the analyticity of $\rho(E)$, it suffices to prove the analyticity of $G(E + i0)$.

The operator H_l will denote the operator H restricted to $l^2(\Lambda_l)$ with Dirichlet boundary conditions. The Green's function corresponding to H_l will be denoted by

$$\begin{aligned} G_l(x, y; z) &= \langle x|\, (H_l - z)^{-1}\, |y\rangle, \qquad x, y \in \Lambda_l, \qquad \mathrm{Im}\, z > 0 \\ G_l(z) &= \mathbf{E}(G_l(0, 0; z)) \end{aligned} \tag{1.6}$$

We will need the following proposition, whose proof will be postponed until Section 5.

Proposition 1.2. If $\mathrm{Im}\, z > 0$, then $\lim_{l \to \infty} G_l(x, y; z) = G(x, y; z)$. In particular, we have $\lim_{l \to \infty} G_l(z) = G(z)$.

In Section 2 we will use the supersymmetric replica trick to rewrite $G_l(z)$ as a two-point function of a supersymmetric field theory. We will show that if h_0 is the characteristic function of the Cauchy distribution with parameter λ, then $G_l(z)$ can be computed explicitly and converges to $G(z)$, which is analytic in z for $\mathrm{Im}\, z > -\lambda$. In Sections 3 and 4 we will examine

$G_l(z)$ in a neighborhood of h_0 in the space S_α with $0<\alpha<\lambda$. Our expression for $G_l(z)$ will give rise to a nonlinear equation and we will then apply an analytic implicit function theorem and a stability theorem to conclude that for any bounded interval I and $0<\alpha'<\alpha$, there exists a neighborhood U around h_0 in S_α such that if $h\in U$, then $G_l(\cdot)$ converges to a function $G(\cdot)$ which is analytic in the region $\{z\in\mathbf{C}\colon \operatorname{Re} z\in I, \operatorname{Im} z> -\alpha'\}$. This will give the analyticity of $\rho(E)$ in the region $\{E\in\mathbf{C}\colon \operatorname{Re} E\in I, |\operatorname{Im} E|<\alpha'\}$. In Section 5 we show that for any h satisfying $h-h_0\in S_\delta$, for any $\delta>(1/2)(k+1)$, we have, for some positive $E_0<+\infty$ and some $\varepsilon>0$, the analyticity of $\rho(E)$ in $\{E\in\mathbf{C}\colon |\operatorname{Re} E|>E_0, |\operatorname{Im} E|<\varepsilon\}$. In Section 6 we will combine the results of Sections 3–5 to obtain Theorem 1.1.

2. THE DENSITY OF STATES FOR THE CAUCHY DISTRIBUTION

The supersymmetric replica trick[5,10] says that if x_1, $x_2\in\Lambda_l$ and $\operatorname{Im} z>0$, then

$$\begin{aligned} G_l(x_1, x_2; z) &= \langle x_1|\,(H_l-z)^{-1}\,|x_2\rangle \\ &= i\int \psi(x_1)\,\bar\psi(x_2)\exp\Big\{-i\sum_{x\in\Lambda_l}\Phi(x)\cdot[(H_l-z)\,\Phi](x)\Big\}\,D_l\Phi \end{aligned} \tag{2.1}$$

where $\Phi(x)=(\varphi(x),\psi(x),\bar\psi(x))$, $\varphi(x)\in\mathbf{R}^2$, $\psi(x)$ and $\bar\psi(x)$ are anticommuting "variables" (i.e., elements of a Grassmann algebra),

$$\Phi(x)\cdot\Phi(y)=\varphi(x)\cdot\varphi(y)+\tfrac{1}{2}[\bar\psi(x)\,\psi(y)+\bar\psi(y)\,\psi(x)]$$

and

$$D_l\Phi=\prod_{x\in\Lambda_l} d\Phi(x)\qquad\text{where}\quad d\Phi(x)=\frac{1}{\pi}\,d\bar\psi(x)\,d\psi(x)\,d^2\varphi(x)$$

To compute functions of ψ, $\bar\psi$, we expand in a power series that terminates after a finite number of terms due to the anticommutativity. The linear functional denoted by integration against $d\bar\psi(x)\,d\psi(x)$ is defined by

$$\int [a_0+a_1\psi(x)+a_2\bar\psi(x)+a_3\bar\psi(x)\,\psi(x)]\,d\bar\psi(x)\,d\psi(x)=-a_3 \tag{2.2}$$

Using the definition of H_l and averaging over the random potential, we obtain

$$G_l(z,h)=\frac{i}{\pi}\int \beta(\varphi^2;z,h)[TB(z,h)\underbrace{(\cdots(TB(z,h)\,1)^k\cdots)^k}_{(l-1)\text{ times}}]^{k+1}(\varphi^2)\,d^2\varphi \tag{2.3}$$

where the dependence on h is indicated and where $\varphi \in \mathbf{R}^2$, $\beta(\varphi^2; z, h) = h(\varphi^2)\, e^{iz\varphi^2}$, $B(z, h)$ denotes the operator multiplication by $\beta(\varphi^2; z, h)$, and T is the operator given by

$$(Tf)(\varphi^2) = -\frac{1}{\pi} \int e^{-i\varphi \cdot \varphi'} f'(\varphi'^2)\, d^2\varphi'$$

Note that T preserves the value at 0, i.e., $(Tf)(0) = f(0)$.

If one defines the Hilbert space[5]

$$\mathscr{H}_1 = \left\{ f\colon [0, \infty) \to \mathbf{C} \text{ absolutely continuous} \middle| \; \|f\|^2_{\mathscr{H}_1} = \sum_{k=0}^{1} \|2^k r^{1/2} f^{(k)}(r^2)\|^2_{L^2([0,\infty), dr)} < \infty \right\} \tag{2.4}$$

and the subspace

$$\mathscr{H}_1^0 = \{ f \in \mathscr{H}_1 ; f(0) = 0 \}$$

then one has that T is unitary on $\mathscr{H}_1$ and $\mathscr{H}_1^0$.

For the case when the potential distribution is the Cauchy distribution with parameter λ, it is known that $G(z) = G^{\text{free}}(z + i\lambda)$, where $G^{\text{free}}(z)$ is the Green's function of H_0 (e.g., refs. 10 and 18). In particular, we get the well-known result $N(E) = N^{\text{free}}(E + i\lambda)$. Thus, we proceed to compute $G^{\text{free}}(z)$.

For the free Hamiltonian H_0, $V(x) = 0$, so $h \equiv 1$. Then

$$\xi_1(\varphi^2) \equiv (TB(z, h)\, 1)^k\, (\varphi^2) = e^{i(-k/4z)\varphi^2} = e^{i\theta_1(z)\varphi^2} \tag{2.5}$$

where

$$\theta_1(z) = \frac{-k}{4z} \tag{2.6}$$

So inductively one gets

$$\xi_n(\varphi^2) = \underbrace{(TB(z, h)(\cdots (TB(z, h)\, 1)^k \cdots)^k)^k}_{n \text{ times}}\, (\varphi^2) = e^{i\theta_n(z)\varphi^2} \tag{2.7}$$

where

$$\theta_n(z) = \frac{-k}{4[z + \theta_{n-1}(z)]} \tag{2.8}$$

Now if θ_n converges to some γ, then we must have

$$\gamma = \frac{-k}{4(z+\gamma)}$$

or

$$\gamma = \tfrac{1}{2}[-z \pm (z^2-k)^{1/2}]$$

Proposition 2.1. For each fixed z with $\operatorname{Im} z > 0$, $\theta_n(z)$ converges to

$$\gamma_+(z) \equiv \tfrac{1}{2}[-z + (z^2-k)^{1/2}]$$

where $\operatorname{Im} \sqrt{\ } > 0$.

Proof. For fixed z with $\operatorname{Im} z > 0$, let

$$v = S(u) = \frac{-k}{4(u+z)}$$

The fixed points of S are

$$\gamma_+(z) = \tfrac{1}{2}[-z + (z^2-k)^{1/2}]$$
$$\gamma_-(z) = \tfrac{1}{2}[-z - (z^2-k)^{1/2}]$$

Also

$$S(0) = \frac{-k}{4z}$$

So S takes γ_+, γ_-, 0 to γ_+, γ_-, $-k/4z$. Recall that if $w = T(z)$ is a fractional linear transformation taking z_1, z_2, z_3 to w_1, w_2, w_3, respectively, then

$$\frac{w-w_1}{w-w_2} \cdot \frac{w_3-w_2}{w_3-w_1} = \frac{z-z_1}{z-z_2} \cdot \frac{z_3-z_2}{z_3-z_1}$$

Thus

$$\begin{aligned}
\frac{v-\gamma_+}{v-\gamma_-} &= \left(\frac{(-k/4z)-\gamma_+}{(-k/4z)-\gamma_-}\right)\left(\frac{0-\gamma_-}{0-\gamma_+}\right)\left(\frac{u-\gamma_+}{u-\gamma_-}\right) \\
&= \left(\frac{\gamma_-}{\gamma_+}\right)\left(\frac{(k/4z)+\gamma_+}{(k/4z)+\gamma_-}\right)\left(\frac{u-\gamma_+}{u-\gamma_-}\right) \\
&= c\left(\frac{u-\gamma_+}{u-\gamma_-}\right)
\end{aligned}$$

where

$$c=\left(\frac{\gamma_-}{\gamma_+}\right)\left(\frac{(k/4z)+\gamma_+}{(k/4z)+\gamma_-}\right)=\frac{(k\gamma_-/4z)+\gamma_+\gamma_-}{(k\gamma_+/4z)+\gamma_+\gamma_-}$$

$$=\frac{(k\gamma_-/4z)+(k/4)}{(k\gamma_+/4z)+(k/4)}$$

$$=\frac{\gamma_-+z}{\gamma_++z}=\frac{z-(z^2-k)^{1/2}}{z+(z^2-k)^{1/2}}=\frac{k}{[z+(z^2-k)^{1/2}]^2}$$

If

$$v_n=\underbrace{S(\cdots(S(u))\cdots)}_{n\text{ times}}$$

then we have

$$\frac{v_n-\gamma_+}{v_n-\gamma_-}=c^n\left(\frac{u-\gamma_+}{u-\gamma_-}\right)$$

Now $|c|^{-1}=(1/k)\,|z+(z^2-k)^{1/2}|^2$ and so we look at the mapping $z\to z+(z^2-k)^{1/2}$.

From Fig. 1 we see that for Im $z>0$, $|z+(z^2-k)^{1/2}|>\sqrt{k}$, so $|c|<1$. As $n\to\infty$, $|c|^n\to 0$, so

$$\left|\frac{v_n-\gamma_+}{v_n-\gamma_-}\right|\to 0$$

which means $v_n\to\gamma_+$. But

$$v_n=\underbrace{S(\cdots(S(u))\cdots)}_{n\text{ times}}$$

which implies

$$v_n=\frac{-k}{4(v_{n-1}+z)}$$

If we take $u=0$, then $v_1=-(k/4z)=\theta_1(z)$ and so $v_n=\theta_n(z)$, which gives us $\theta_n(z)\to\gamma_+(z)$. ∎

Now we also have

$$\xi_n(\varphi^2)=\underbrace{(TB(z,h)(\cdots(TB(z,h)\,1)^k\cdots)^k)^k}_{n\text{ times}}(\varphi^2)$$

$$=e^{i\theta_n(z)\varphi^2}\to e^{i\gamma_+(z)\varphi^2}\equiv f_0(\varphi^2)\tag{2.9}$$

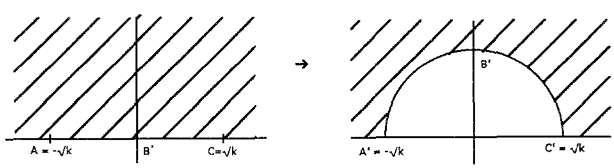

Fig. 1. The map $z\to z+(z^2-k)^{1/2}$.

So by the dominated convergence theorem we have

$$G^{\text{free}}(z) = \lim_{l\to\infty} G_l^{\text{free}}(z) = \frac{i}{\pi}\int e^{iz\varphi^2} e^{i\gamma_+(z)\varphi^2} e^{i(1/k)\gamma_+(z)\varphi^2}\, d^2\varphi$$

$$= \frac{-1}{z + [1+(1/k)]\,\gamma_+(z)} \tag{2.10}$$

for $\operatorname{Im} z > 0$. It can easily be seen that $G^{\text{free}}(E+i0)$ exists for all E and so $N^{\text{free}}(E)$ is absolutely continuous and the density of states is then given by

$$\rho^{\text{free}}(E) = \begin{cases} \dfrac{2}{\pi}\dfrac{(k+1)(k-E^2)^{1/2}}{(k+1)^2 - 4E^2} & \text{if} \quad E^2 < k \\ 0 & \text{otherwise} \end{cases} \tag{2.11}$$

In addition, we see from (2.11) that the spectrum of H_0, $\sigma(H_0)$, is $[-\sqrt{k}, +\sqrt{k}]$.

For the Cauchy distribution with parameter $\lambda > 0$, we have $G(z) = G^{\text{free}}(z + i\lambda)$. Thus

$$G(z) = \frac{-1}{z + i\lambda + [1+(1/k)]\,\gamma_+(z+i\lambda)} \tag{2.12}$$

which yields

$$\rho(E) = \frac{1}{\pi}\operatorname{Im}\frac{-1}{E+i\lambda + [1+(1/k)]\{-(E+i\lambda) + [(E+i\lambda)^2 - k]^{1/2}\}} \tag{2.13}$$

So we see from (2.13) that for the Cauchy distribution, $\sigma(H) = \mathbf{R}$ and the density of states is analytic in the strip $|\operatorname{Im} E| < \lambda$.

3. CONVERGENCE OF THE AVERAGED GREEN'S FUNCTION FOR DISTRIBUTIONS NEAR THE CAUCHY DISTRIBUTION

Recall from Section 2 that we have for $\operatorname{Im} z > 0$

$$G_l(z,h) = \frac{i}{\pi}\int \beta(\varphi^2; z, h)[(TB(z,h)\underbrace{(\cdots(TB(z,h)\,1)^k\cdots)^k}_{(l-1)\text{times}}]^{k+1}(\varphi^2)\, d^2\varphi \tag{3.1}$$

If we let

$$g(z,h,f) = (TB(z,h)\,f)^k \tag{3.2}$$

then (3.1) can be rewritten as

$$G_l(z, h) = \frac{i}{\pi} \int \beta(\varphi^2; z, h)[g^l(z, h, 1)(TB(z, h)\, g^{l-1}(z, h, 1))](\varphi^2)\, d^2\varphi \quad (3.3)$$

where

$$g^l(z, h, 1) = \underbrace{[TB(z, h)(\cdots (TB(z, h)\, 1)^k \cdots)^k]^k}_{l \text{ times}} \quad (3.4)$$

For the case where the distribution is the Cauchy distribution, i.e., $h_0(\varphi^2) = e^{-\lambda\varphi^2}$, we know that $g^l(z, h_0, 1)$ converges to the fixed point f_0, where

$$f_0(\varphi^2) = e^{i\gamma_+(z + i\lambda)\varphi^2}$$

and

$$\gamma_+(z) = \tfrac{1}{2}[-z + (z^2 - k)^{1/2}] \qquad (\operatorname{Im} \surd > 0)$$

It will be shown that for distributions sufficiently close to the Cauchy distribution, $g^l(z, h, 1)$ converges to a fixed point f which is close to f_0.

Recall the Banach space

$$S_\alpha = \left\{ f\colon [0, \infty) \to \mathbf{C} \text{ absolutely continuous} \mid \|f\|_{S_\alpha} = \sum_{k=0}^{1} \|e^{\alpha t} f^{(k)}(t)\|_\infty < \infty \right\}$$

where α is fixed and is in the interval $(0, \lambda)$. Now define the Banach spaces

$$\begin{aligned} \bar{\mathscr{H}}_1 &= \left\{ f \in \mathscr{H}_1 \colon \|f\|^2_{\bar{\mathscr{H}}_1} = \|f\|^2_{\mathscr{H}_1} + \sum_{k=0}^{1} \|f^{(k)}\|^2_\infty < +\infty \right\} \\ \bar{\mathscr{H}}^0_1 &= \{ f \in \bar{\mathscr{H}}_1 \colon f(0) = 0 \} \end{aligned} \quad (3.5)$$

Let $D_\alpha = \{z \in \mathbf{C}\colon \operatorname{Im} z > -\alpha\}$. The first thing we need to do is show that g given by (3.2) is a continuous mapping from $D_\alpha \times S_\alpha \times \bar{\mathscr{H}}_1$ into $\bar{\mathscr{H}}_1$, i.e., $g(z, h, f) \in \bar{\mathscr{H}}_1$ for all $(z, h, f) \in D_\alpha \times S_\alpha \times \bar{\mathscr{H}}_1$. Noting that $\bar{\mathscr{H}}_1$ forms an algebra, it suffices to show that the mapping $(z, h, f) \to TB(z, h) f$ is a continuous mapping from $D_\alpha \times S_\alpha \times \bar{\mathscr{H}}_1$ into $\bar{\mathscr{H}}_1$. It is straightforward to show the continuity of this map. Thus we need only show that $TB(z, h) f \in \bar{\mathscr{H}}_1$.

Now[10]

$$\begin{aligned} (Tf)(\varphi^2) &= -\frac{1}{\pi} \int e^{-i\varphi_1 \cdot \varphi} f'(\varphi_1^2)\, d^2\varphi_1 \\ (Tf)'(\varphi^2) &= -\frac{1}{4\pi} \int e^{-i\varphi_1 \cdot \varphi} f(\varphi_1^2)\, d^2\varphi_1 \end{aligned} \quad (3.6)$$

and so

$$\|TB(z,h)f\|_\infty \leqslant \frac{1}{\pi}\|(\beta f)'\|_{L^1([0,\infty),dr)}$$

$$\leqslant \frac{1}{\pi}\|f\|_\infty\|\beta'\|_{L^1([0,\infty),dr)} + \frac{1}{\pi}\|f'\|_\infty\|\beta\|_{L^1([0,\infty),dr)} < \infty$$

$$\|(TB(z,h)f)'\|_\infty \leqslant \frac{1}{4\pi}\|\beta f\|_{L^1([0,\infty),dr)}$$

$$\leqslant \frac{1}{4\pi}\|f\|_\infty\|\beta\|_{L^1([0,\infty),dr)} < \infty \tag{3.7}$$

Finally, if $f \in \bar{\mathscr{H}}_1$, then $f \in \mathscr{H}_1$ and $B(z,h)f \in \mathscr{H}_1$. Since T is unitary on $\mathscr{H}_1$, it follows that $TB(z,h)f \in \mathscr{H}_1$. So we have that $TB(z,h)f \in \bar{\mathscr{H}}_1$.

It should be noted that although $f(\varphi^2) \equiv 1$ is not in $\bar{\mathscr{H}}_1$, $TB(z,h)$ does map the function which is identically 1 to $\bar{\mathscr{H}}_1$. Thus, $g(z,h,1)$ is in $\bar{\mathscr{H}}_1$ and consequently so is $g^l(z,h,1)$.

We will study the convergence of $\{g^n(z,h,f)\}$, where

$$g^n(z,h,f) = g(z,h,g^{n-1}(z,h,f))$$

If $g^n(z,h,f)$ converges, then the limit must be a solution to the equation

$$f = g(z,h,f) \tag{3.8}$$

Thus we will consider the equation

$$F(z,h,f) = g(z,h,f) - f = 0 \tag{3.9}$$

3.1. THE DIFFERENTIABILITY OF *F*

Proposition 3.1. Let $\operatorname{Im} z_0 > -\alpha$. Then the map $F: D_\alpha \times S_\alpha \times \bar{\mathscr{H}}_1 \to \bar{\mathscr{H}}_1$ given by (3.9) is Frechet differentiable in a neighborhood $W \subset D_\alpha \times S_\alpha \times \bar{\mathscr{H}}_1$ of (z_0, h_0, f_0).

Proof. Let $y = (z,h,f)$, $y_0 = (z_0, h_0, f_0)$, and

$$\|y\|_{\mathbf{C}\times S_\alpha \times \bar{\mathscr{H}}_1} = \max(|z|, \|h\|_{S_\alpha}, \|f\|_{\bar{\mathscr{H}}_1})$$

We will assume throughout this proof that $\|y - y_0\|_{\mathbf{C}\times S_\alpha\times\bar{\mathscr{H}}_1} < \delta$ for some small $\delta > 0$.

Define the map Q: $W \to \bar{\mathscr{H}}_1$ by

$$Q(y)(r) = (B(z,h)f)(r) = \beta(r; z, h)f(r)$$

It is straightforward to show that

$$\|Q(y)-Q(y_0)-Q'(y_0)(y-y_0)\|_{\bar{\mathscr{H}}_1} \leqslant C' \, \|y-y_0\|^2_{\mathbf{C}\times S_\alpha\times\bar{\mathscr{H}}_1} \tag{3.10}$$

where $Q'(y_0)$ is a bounded linear operator from $\mathbf{C}\times S_\alpha\times\bar{\mathscr{H}}_1$ to $\bar{\mathscr{H}}_1$ given by

$$(Q'(y_0)\,y)(r) = irze^{iz_0r}h_0(r)\,f_0(r) + e^{iz_0r}h(r)\,f_0(r) + e^{iz_0r}h_0(r)\,f(r)$$

Here C' is a constant which can be chosen uniformly for all y_0 in a compact set in $D_\alpha\times S_\alpha\times\bar{\mathscr{H}}_1$ (we will need this uniformity later). Thus we have the differentiability of Q in a neighborhood W of y_0.

Now if T were a bounded linear operator on $\bar{\mathscr{H}}_1$, we would immediately have the same type of estimate for $TQ'(y_0)$ as (3.10). Let χ be the operator on $\bar{\mathscr{H}}_1$ which is multiplication by $e^{\varepsilon r}$ with $-\alpha<-\varepsilon<\operatorname{Im} z_0$. Then χQ is differentiable with an estimate as in (3.10) and $T\chi^{-1}$ is a bounded linear operator on $\bar{\mathscr{H}}_1$. It then follows that we will get the same type of estimate as in (3.10) for the derivative of $(T\chi^{-1})(\chi Q)=TQ$ at y_0.

The differentiability of g, where $g(y)=g(z,h,f)=(TB(z,h)\,f)^k$, now follows easily. In fact, we again have

$$\|g(y)-g(y_0)-g'(y_0)(y-y_0)\|_{\bar{\mathscr{H}}_1} \leqslant C'' \, \|y-y_0\|^2_{\mathbf{C}\times S_\alpha\times\bar{\mathscr{H}}_1} \tag{3.11}$$

where, as before, C'' is a constant which can be chosen uniformly for all y_0 in a compact set in $D_\alpha\times S_\alpha\times\bar{\mathscr{H}}_1$. ∎

3.2. The Spectrum of the Linearized Operator $F_f(z_0, h_0, f_0)$

We will now examine the spectrum of the derivative $F_f(z_0,h_0,f_0)$. Our goal here is to show that $F_f(z_0,h_0,f_0)$ is nonsingular. Since $F_f(z_0,h_0,f_0)=g_f(z_0,h_0,f_0)-I$, we need only show that

$$g_f(z_0,h_0,f_0)=k(TB(z_0,h_0)\,f_0)^{k-1}\,TB(z_0,h_0)$$

does not contain 1 in its spectrum.

Let

$$f_t(\varphi^2)=e^{t\varphi^2}f_0(\varphi^2)$$

Then for $|t|$ small

$$(TB(z_0,h_0)\,f_t)(\varphi^2)=\exp\{-i[4(z_0+i\lambda+\gamma_+(z_0+i\lambda)-it)]^{-1}\,\varphi^2\} \tag{3.12}$$

where $\operatorname{Im} z_0>-\lambda$. Then we have

$$[(TB(z_0,h_0)\,f_0)^{k-1}\,TB(z_0,h_0)\,f_t](\varphi^2)=e^{r(t)\varphi^2}f_0(\varphi^2)=f_{r(t)}(\varphi^2) \tag{3.13}$$

where

$$r(t)=i\left(\frac{-\gamma_+(z_0+i\lambda)}{k}-\frac{1}{4[z_0+i\lambda+\gamma_+(z_0+i\lambda)-it]}\right)$$
$$=i\left(\frac{-\gamma_+(z_0+i\lambda)}{k}-\frac{1}{4[q(z_0+i\lambda)-it]}\right) \tag{3.14}$$

and

$$q(z)=z+\gamma_+(z) \tag{3.15}$$

If $A=k[TB(z_0,h_0)f_0]^{k-1}\,TB(z_0,h_0)$, then

$$Af_t(\varphi^2)=kf_{r(t)}(\varphi^2) \tag{3.16}$$

and so

$$A(\varphi^{2n}f_0)=A\frac{\partial^n}{\partial t^n}f_t|_{t=0}$$
$$=\frac{\partial^n}{\partial t^n}Af_t|_{t=0}$$
$$=k\frac{\partial^n}{\partial t^n}f_{r(t)}|_{t=0} \tag{3.17}$$

We can now prove the following about the spectrum of the operator $g_f(z_0,h_0,f_0)$ on $\mathscr{H}_1$ where $\operatorname{Im} z_0>-\lambda$:

Proposition 3.2. If $\operatorname{Im} z_0>-\lambda$, the operator

$$k[TB(z_0,h_0)f_0]^{k-1}\,TB(z_0,h_0)\colon \mathscr{H}_1\to\mathscr{H}_1$$

has eigenvectors of the form $(\varphi^{2n}+a_{n-1}\varphi^{2(n-1)}+\cdots+a_0)f_0$ for every $n=0,1,2,\ldots$ and $a_0=0$ for $n\geqslant 1$. The corresponding eigenvalues are $E_n=k[2q(z_0+i\lambda)]^{-2n}$, so that $|E_n|<1$ for $n\geqslant 1$ and $E_n\to 0$ as $n\to+\infty$.

Proof. Equation (3.16) tells us that

$$k[TB(z_0,h_0)f_0]^{k-1}\,TB(z_0,h_0)f_0=kf_0$$

So for the case $n=0$ we are done. We now seek an eigenvector of the form $(\varphi^2+a)f_0$. It must satisfy the equation

$$A(\varphi^2+a)f_0=E_1(\varphi^2+a)f_0 \tag{3.18}$$

Now

$$\frac{\partial}{\partial t} f_{r(t)} = \frac{\partial f_{r(t)}}{\partial r} r'(t) = \varphi^2 r'(t)\, e^{r(t)\varphi^2} f_0(\varphi^2) \tag{3.19}$$

and Eq. (3.14) gives $r'(t) = [4(q-it)^2]^{-1}$ and $r'(0) = (4q^2)^{-1}$ [here $q = q(z_0 + i\lambda)$]. So using Eq. (3.17) with $n = 1$ gives

$$A(\varphi^2 f_0) = k \frac{\partial}{\partial t} f_{r(t)} \bigg|_{t=0} = \frac{k}{4q^2} \varphi^2 f_0 \tag{3.20}$$

Substituting (3.20) into (3.18) gives

$$k\left(\frac{\varphi^2}{4q^2} + a\right) f_0 = (E_1 \varphi^2 + E_1 a) f_0 \tag{3.21}$$

which implies $E_1 = k/4q^2$ and $a = 0$. Since $q(z) = z + \gamma_+(z)$, it follows that $2q(z) = z + (z^2 - k)^{1/2}$. From Fig. 1 we see that for $\operatorname{Im}(z_0 + i\lambda) > 0$, $|2q| > \sqrt{k}$, or, equivalently, $|E_1| < 1$. So $A(\varphi^2 f_0) = E_1 \varphi^2 f_0$ with $|E_1| = |k/4q^2| < 1$.

For $n \geqslant 2$ we have

$$\frac{\partial^n}{\partial t^n} f_{r(t)} = [\varphi^{2n}(r'(t))^n + \cdots + \varphi^2 r^{(n)}(t)]\, e^{r(t)\varphi^2} f_0(\varphi^2) \tag{3.22}$$

and so

$$A(\varphi^{2n} + \cdots + a_0) f_0(\varphi^2) = [k(4q^2)^{-n} \varphi^{2n} + \cdots + k a_0] f_0(\varphi^2) \tag{3.23}$$

The eigenvector equation is

$$[k(4q^2)^{-n} \varphi^{2n} + \cdots + k a_0)] f_0(\varphi^2) = E_n(\varphi^{2n} + \cdots + a_0) f_0(\varphi^2) \tag{3.24}$$

so that $E_n = k(4q^2)^{-n}$ and we can solve for the coefficients $a_{n-1},\ldots, a_0$. Since (3.24) implies that $k a_0 = E_n a_0$, we must have $a_0 = 0$. ▌

Proposition 3.3. The set of eigenvectors in Proposition 3.2 form a complete set in $\mathscr{H}_1$.

Proof. Consider the isometry

$$\pi = 2^{-1/2}(1 - 2\partial)\colon \tilde{\mathscr{H}}_1^0 \to \tilde{L}^2([0, \infty), dr)$$

where $\partial f(r) = f'(r)$, $\tilde{\mathscr{H}}_1^0$ is $\mathscr{H}_1^0$ with real inner product $\operatorname{Re}\langle \cdot, \cdot \rangle_{\mathscr{H}_1^0}$, and $\tilde{L}^2([0, \infty), dr)$ is $L^2([0, \infty), dr)$ with real inner product $\operatorname{Re}\langle \cdot, \cdot \rangle_{L^2([0,\infty),dr)}$. That π is an isometry follows from

$$\begin{aligned}\langle f, g\rangle_{\mathscr{H}_1^0} &= \tfrac{1}{2}\langle f, (1-4\partial^2)\, g\rangle_{\tilde{L}^2([0,\infty), dr)} \\ &= \tfrac{1}{2}\langle (1-2\partial) f, (1-2\partial)\, g\rangle_{\tilde{L}^2([0,\infty), dr)} \\ &= \langle \pi(f), \pi(g)\rangle_{\tilde{L}^2([0,\infty), dr)} \end{aligned} \tag{3.25}$$

Note that for functions f and g in $L^2(\mathbf{R}, dr)$ which obey $f(-r)=\bar{f}(r)$ and similarly for g, we have

$$\operatorname{Re}\langle f, g\rangle_{L^2([0,\infty), dr)} = \langle f, g\rangle_{L^2(R, (1/2)dr)}$$

Also note that π is not onto, e.g., no function in $\mathscr{H}_1^0$ is taken to $\exp(i\gamma_+ r)$.

Let $K=\{r^n e^{i\gamma_+ r},\ n=1, 2,...\}$. We will need the following result.

Lemma 3.4. $K\cup\{\exp(i\gamma_+ r\}$ is complete in $\tilde{L}^2([0,\infty), dr)$.

Proof. Suppose

$$\langle \eta, r^m e^{i\gamma_+ r}\rangle_{\tilde{L}^2([0,\infty), dr)} = 0 \qquad \text{for every} \quad m=0, 1, 2,...$$

For $\gamma_+ = s+it$, $t>0$, we have

$$\begin{aligned}\langle \eta, e^{ikr} e^{i\gamma_+ r}\rangle_{\tilde{L}^2([0,\infty), dr)} &= \langle \eta, e^{-tr} e^{i(k+s)r}\rangle_{\tilde{L}^2([0,\infty), dr)} \\ &= \tfrac{1}{2}\widehat{\bar{\eta}\psi}(k+s)\end{aligned} \tag{3.26}$$

where $\widehat{\ }$ is the Fourier transform in $L^2(\mathbf{R})$, $\psi(r)=e^{-t|r|}$, and η is extended so it satisfies $\eta(-r)=\bar{\eta}(r)$. Also for $k\in\mathbf{R}$ and $|k|$ small we have

$$\langle \eta, e^{ikr} e^{i\gamma_+ r}\rangle_{\tilde{L}^2([0,\infty), dr)} = \sum_{m=0}^{\infty} \frac{(ik)^m}{m!}\langle \eta, r^m e^{i\gamma_+ r}\rangle_{\tilde{L}^2([0,\infty), dr)} = 0 \tag{3.27}$$

Then (3.26) and (3.27) imply that $\eta=0$. Thus, the orthogonal complement of the set $K\cup\{e^{i\gamma_+ r}\}$ is 0, so $K\cup\{e^{i\gamma_+ r}\}$ is complete in $\tilde{L}^2([0,\infty), dr)$. ∎

We can now complete the proof of Proposition 3.3. Now $span(K\cup\{e^{i\gamma_+ r}\}) = span(\pi(K)\cup\{e^{i\gamma_+ r}\})$. Since π is an isometry, $\{e^{i\gamma_+ r}\}$ is linearly independent of K, and $K\cup\{e^{i\gamma_+ r}\}$ is complete in $\tilde{L}^2([0,\infty), dr)$, it follows that $\pi(span\ K)$ has codimension 1 in $\tilde{L}^2([0,\infty), dr)$. Also, since

$$\pi(span\ K)\subset \pi(\mathscr{H}_1^0)\subset \tilde{L}^2([0,\infty), dr)$$

and

$$\pi(\mathscr{H}_1^0)\neq \tilde{L}^2([0,\infty), dr)$$

we have that $\pi(span\ K)=\pi(\mathscr{H}_1^0)$. Thus, K must be complete in $\mathscr{H}_1^0$.

Since $\tilde{\mathscr{H}}_1$ can be decomposed as $\tilde{\mathscr{H}}_1 = \mathbf{C}e^{i\gamma+r} \oplus \tilde{\mathscr{H}}_1^0$, it follows that $K \cup \{e^{i\gamma+r}\}$ is complete in $\tilde{\mathscr{H}}_1$. The inner product is completely determined by its real part, so we have that $K \cup \{e^{i\gamma+r}\}$ is complete in $\mathscr{H}_1$. Thus, our eigenvectors form a complete set in $\mathscr{H}_1$. ∎

So we have proven the following result.

Theorem 3.5. The spectrum of the operator $g_f(z_0, h_0, f_0)$ on $\mathscr{H}_1$ consists solely of the eigenvalues $E_n = k[2q(z+i\lambda)]^{-2n}$ with $n = 0, 1, 2,...$ and the operator $g_f(z_0, h_0, f_0)$ restricted to $\mathscr{H}_1^0$ has spectral radius strictly less than 1.

We have established that $g_f(z_0, h_0, f_0)$ restricted to $\mathscr{H}_1^0$ has spectral radius less than 1. We would like to show that this property holds for $g_f(z_0, h_0, f_0)$ restricted to $\tilde{\mathscr{H}}_1^0$. Let B_1 and B_2 be the operators multiplication by β_1 and β_2, respectively. Klein and Speis[12] and also March and Sznitman[6] have proved the following result.

Lemma 3.6. Let $\{f_n\}_{n\in\mathbf{N}}$ be a sequence in $\mathscr{H}_1$ such that $\|f_n\|_{\mathscr{H}_1} \leqslant M$ for all $n \in \mathbf{N}$ and some $M < +\infty$. Also let β_1 and β_2 be such that $\|e^{\alpha_{ik}r}\beta_i^{(k)}(r)\|_\infty < C$ for some $C < +\infty$, $\alpha_{ik} > 0$, $i = 1, 2$, and $k = 0, 1$. Then there exists a subsequence $\{f_{n_i}\}_{i\in\mathbf{N}}$ such that $\{B_2 T B_1 f_{n_i}\}_{i\in\mathbf{N}}$ is Cauchy in $\mathscr{H}_1$.

We will now show that Lemma 3.6 holds in $\tilde{\mathscr{H}}_1$.

Lemma 3.7. Let $\{f_n\}_{n\in\mathbf{N}}$ be a sequence of elements in $C(\mathbf{R}^q)$ such that $\|f_n\|_\infty \leqslant M$ for all $n \in \mathbf{N}$ and some $M < +\infty$. Also let β_1 and β_2 be in $C_0(\mathbf{R}^q)$, the space of continuous functions on $\mathbf{R}^q$ with compact support. If $\mathscr{F}$ is the usual Fourier transform in $\mathbf{R}^q$, then there exists a subsequence $\{f_{n_i}\}_{i\in\mathbf{N}}$ such that $\{B_2 \mathscr{F} B_1 f_{n_i}\}_{i\in\mathbf{N}}$ is Cauchy in $C(\mathbf{R}^q)$.

Proof. Fix n. Then

$$\mathscr{F}[\beta_1 f_n](y) = (2\pi)^{-q/2} \int e^{ix\cdot y}\beta_1(x) f_n(x)\, dx$$

$$= (2\pi)^{-q/2} \int_{D_1} e^{ix\cdot y}\beta_1(x) f_n(x)\, dx$$

where D_1 is the support of β_1.

Now

$$|\mathscr{F}(\beta_1 f_n)(y) - \mathscr{F}(\beta_1 f_n)(y')| \leqslant C|y - y'| \tag{3.28}$$

Also

$$|\mathscr{F}(\beta_1 f_n)(y)| \leqslant C\, \|\beta_1 f_n\|_{L^1([0,\infty), dr)} \leqslant CM\, \|\beta_1\|_{L^1([0,\infty), dr)} = C'' \tag{3.29}$$

Thus $\{\mathscr{F}B_1 f_n\}_{n\in\mathbf{N}}$ is a family of bounded equicontinuous functions. Since β_2 has compact support, it follows by the Arzela–Ascoli theorem that there exists a subsequence $\{f_{n_i}\}_{i\in\mathbf{N}}$ such that $\{B_2\mathscr{F}B_1 f_{n_i}\}_{i\in\mathbf{N}}$ is a Cauchy sequence in $C(\mathbf{R}^q)$. ∎

Lemma 3.8. Let $\{f_n\}_{n\in\mathbf{N}}$ be a sequence in $\bar{\mathscr{H}}_1$ such that $\|f_n\|_{\bar{\mathscr{H}}_1}\leqslant M$ for all $n\in\mathbf{N}$ and some $M<+\infty$. Suppose β_1 and β_2 are such that $\|e^{\alpha_{ik}r}\beta_i^{(k)}(r)\|_\infty<C$ for some $C<+\infty$, $\alpha_{ik}>0$, $i=1,2$, and $k=0,1$. Then there exists a subsequence $\{f_{n_i}\}_{i\in N}$ such that $\{B_2TB_1 f_{n_i}\}_{i\in\mathbf{N}}$ is Cauchy in $\bar{\mathscr{H}}_1$.

Proof. We first assume that β_1 and β_2 have compact support. Using Leibniz' rule and (3.7), we have

$$[(B_2TB_1)f](\varphi^2)=-\frac{1}{\pi}\sum_{k_1+k_2=1}\beta_2(\varphi^2)\int e^{-i\varphi'\cdot\varphi}\beta_1^{(k_1)}(\varphi'^2)f^{(k_2)}(\varphi'^2)\,d^2\varphi'$$

$$\begin{aligned}[(B_2TB_1)f]'(\varphi^2)=&-\frac{1}{4\pi}\beta_2(\varphi^2)\int e^{-i\varphi'\cdot\varphi}\beta_1(\varphi'^2)f(\varphi'^2)\,d^2\varphi'\\&-\frac{1}{\pi}\sum_{k_1+k_2=1}\beta_2'(\varphi^2)\int e^{-i\varphi'\cdot\varphi}\beta_1^{(k_1)}(\varphi'^2)f^{(k_2)}(\varphi'^2)\,d^2\varphi'\end{aligned}\tag{3.30}$$

From Lemma 3.6, Lemma 3.7, and the definition of $\bar{\mathscr{H}}_1$, we get that there exists a subsequence $\{f_{n_i}\}_{i\in\mathbf{N}}$ of $\{f_n\}_{n\in\mathbf{N}}$ such that $\{B_2TB_1 f_{n_i}\}_{i\in\mathbf{N}}$ is Cauchy in $\bar{\mathscr{H}}_1$.

Let $\{g_q\}_{q\in\mathbf{N}}$ be a family of real-valued functions defined on $[0,\infty)$ with the following properties:

(1) g_q is of class $C^\infty([0,\infty))$ and for some $M<+\infty$,

$$|g_q^{(k)}(r)|\leqslant M\qquad\text{for all}\quad q\in\mathbf{N},\quad r\in[0,\infty),\quad\text{and}\quad k=0,1$$

(2) $g_q|_{[0,q)}=1$ and $g_q|_{[q+1,\infty)}=0$ for all $q\in\mathbf{N}$.

Let G_q denote the operator multiplication by g_q, $q\in\mathbf{N}$. If $f\in\bar{\mathscr{H}}_1$, then

$$\begin{aligned}\|G_qB_1f-B_1f\|_\infty&=\|(g_q-1)\,\beta_1f\|_\infty\\ \|(G_qB_1f)'-(B_1f)'\|_\infty&=\|(g_q-1)(\beta_1f)'\|_\infty=\|g_q'\beta_1f\|_\infty\end{aligned}\tag{3.31}$$

But $\beta_1^{(k)}g_q\to\beta_1^{(k)}$ in the sup norm as $q\to+\infty$ for $k=0,1$ and $g_q'\beta_1\to 0$ in the sup norm as $q\to+\infty$. So, again using Lemma 3.6 and the definition of $\bar{\mathscr{H}}_1$, we have the lemma. ∎

Lemma 3.8 gives the following result.

Proposition 3.9. The operator $g_f(z_0, h_0, f_0)$ restricted to $\bar{\mathscr{H}}_1^0$ has spectral radius strictly less than 1.

Proof. From Lemma 3.8 we have that $g_f(z_0, h_0, f_0)$ is a compact operator on $\bar{\mathscr{H}}_1^0$. Thus we know that the spectrum of $g_f(z_0, h_0, f_0)$ is a discrete set having no limit points except possibly 0 and any element in the spectrum of $g_f(z_0, h_0, f_0)$ is an eigenvalue of finite multiplicity. Since $\bar{\mathscr{H}}_1^0 \subset \mathscr{H}_1^0$, every eigenvalue of $\bar{\mathscr{H}}_1^0$ must be an eigenvalue of $\mathscr{H}_1^0$. From Proposition 3.3 all of the eigenvalues of $\mathscr{H}_1^0$ are of the form $E_n = k(1/4q^2)^n$ with $n \geqslant 1$. Since $|E_n| < 1$ for $n \geqslant 1$, it follows that the spectral radius of $g_f(z_0, h_0, f_0)$ is strictly less than 1 when restricted to $\bar{\mathscr{H}}_1^0$. ∎

3.3. Stability of the Fixed Point f_0

We now introduce some ideas from nonlinear analysis (see, e.g., Berger[13] and Hirsch and Smale[14]).

We will need the following well-known result.

Lemma 3.10. If there exists an operator $A: W \to B$ such that the spectral radius of A, $\rho(A)$, is less than 1, then there exists an equivalent norm $\|\cdot\|_e$ with $\|Af\|_e \leqslant \mu \|f\|_e$ for some $\mu < 1$ and for all $f \in B$.

Proof. Now $\rho(A) = \lim_{n\to\infty} \|A\|^{1/n} = \lambda < 1$, so there exists some N such that for $n \geqslant N$, $\|A^n\| \leqslant \rho^n < 1$. So $\sum_{n=0}^{\infty} \|A^n\| < \infty$. Define

$$\|f\|_e = \sum_{n=0}^{\infty} \|A^n f\| \tag{3.32}$$

Clearly $\|\cdot\|_e$ is a norm. We want to show the following:

(i) $C_1 \|f\| \leqslant \|f\|_e \leqslant C_2 \|f\|$ for some $C_1, C_2 > 0$.

(ii) $\|Af\|_e \leqslant \mu \|f\|_e$ for some $\mu < 1$ and all $f \in B$.

Since

$$\|f\| \leqslant \|f\|_e = \sum_{n=0}^{\infty} \|A^n f\| \tag{3.33}$$

and

$$\sum_{n=0}^{\infty} \|A^n f\| \leqslant \left(\sum_{n=0}^{\infty} \|A^n\|\right) \|f\| = C_2 \|f\| \tag{3.34}$$

where $C_2 = \sum_{n=0}^{\infty} \|A^n\|$, we have (i) with $C_1 = 1$. Now

$$\|Af\|_e = \sum_{n=1}^{\infty} \|A^n f\| = \|f\|_e - \|f\| \tag{3.35}$$

so

$$\frac{\|Af\|_e}{\|f\|_e} = 1 - \frac{\|f\|}{\|f\|_e} \leqslant 1 - \frac{1}{C_2} \quad \blacksquare \tag{3.36}$$

Proposition 3.11. There exists a neighborhood V around f_0 such that $g^n(z_0, h_0, f)$ converges to f_0 for every $f \in V$ with $f(0) = 1$.

Proof. Since $g_f(z_0, h_0, f_0)$ has spectral radius less than 1 when restricted to $\mathscr{H}_1^0$, we can define an equivalent norm on $\mathscr{H}_1^0$ by

$$\|f\|_e = \sum_{n=0}^{\infty} \|g_f^n(z_0, h_0, f_0) f\|_{\mathscr{H}_1} \tag{3.37}$$

and recall that there exists some $\mu < 1$ such that

$$\|g_f(z_0, h_0, f_0) f\|_e \leqslant \mu \|f\|_e \tag{3.38}$$

for all $f \in \mathscr{H}_1^0$ (Lemma 3.10).

Let $0 < \varepsilon < 1 - \mu$ and let C_1 and C_2 be positive constants such that

$$C_1 \|f\|_e \leqslant \|f\|_{\mathscr{H}_1} \leqslant C_2 \|f\|_e \tag{3.39}$$

for all $f \in \mathscr{H}_1^0$. From the definition of the derivative there exist a neighborhood $V \subset \mathscr{H}_1$ of f_0 so small that

$$\begin{aligned} &\|g(z_0, h_0, f) - g(z_0, h_0, f_0) - g_f(z_0, h_0, f_0)(f - f_0)\|_{\mathscr{H}_1} \\ &\quad \leqslant \varepsilon(C_1/C_2) \|f - f_0\|_{\mathscr{H}_1} \end{aligned} \tag{3.40}$$

for every $f \in V$. Since $f(0) = (TB(z, h) f)(0)$ and $g(z_0, h_0, f) = (TB(z_0, h_0) f)^k$, it follows that if $f(0) = 1$, then both $g(z_0, h_0, f) - g(z_0, h_0, f_0)$ and $f - f_0$ will belong to $\mathscr{H}_1^0$. So, using (3.39) in (3.40) gives

$$\|g(z_0, h_0, f) - g(z_0, h_0, f_0) - g_f(z_0, h_0, f_0)(f - f_0)\|_e \leqslant \varepsilon \|f - f_0\|_e \tag{3.41}$$

Thus

$$\begin{aligned} \|g(z_0, h_0, f) - g(z_0, h_0, f_0)\|_e &\leqslant \|g_f(z_0, h_0, f_0)(f - f_0)\|_e + \varepsilon \|f - f_0\|_e \\ &\leqslant \mu \|f - f_0\|_e + \varepsilon \|f - f_0\|_e \\ &= \nu \|f - f_0\|_e \end{aligned} \tag{3.42}$$

where $\nu = \mu + \varepsilon < 1$. Using the fact that $g(z_0, h_0, f_0) = f_0$, we have

$$\|g^n(z_0, h_0, f) - f_0\|_e \leqslant \nu^n \|f - f_0\|_e \tag{3.43}$$

and so $g^n(z_0, h_0, f)$ converges to f_0 for all $f \in V$ with $f(0) = 1$. $\blacksquare$

3.4. Convergence of $G_l(z, h)$ in a Neighborhood of (z_0, h_0)

We now introduce the notion of an analytic mapping (see Berger,[13] p. 84).

Definition 3.12. Let X and Y be Banach spaces over the complex numbers, and let U be a connected open subset of X. Then the mapping f from U into Y is complex analytic if for each $x \in U$, $h \in X$, $y^* \in Y^*$ (Y^* being the dual space of Y), $y^*(f(x+th))$ is an analytic function of the complex variable t for $|t|$ sufficiently small.

Note that for the case $X = Y = \mathbf{C}$, the usual definition of an analytic function is implied.

Now if F is Frechet differentiable in a neighborhood W around (z_0, h_0, f_0), then F is analytic in W (see Berger,[13] pp. 84–88). Recall that $F_f(z_0, h_0, f_0) = g_f(z_0, h_0, f_0) - I$ and the spectrum of $g_f(z_0, h_0, f_0)$ does not contain 1. Thus $F_f(z_0, h_0, f_0)$ is invertible and we can apply an analytic implicit function theorem (see Berger,[13] p. 134) to conclude the existence of an analytic mapping $f: U \to \mathscr{H}_1$ which is the unique solution to $F(z, h, f(z, h)) = 0$ in a neighborhood U around (z_0, h_0) with $f(z_0, h_0) = f_0$. Note that since $TB(z, h)$ preserves the value at 0, we must have $f(z, h)(0) = 1$.

Since the mapping $(z, h) \to f(z, h)$ is continuous, it follows that there exists a neighborhood $U' \subset U$ around (z_0, h_0) such that if $(z, h) \in U'$, then $f(z, h) \in V$. The mapping $(z, h) \to g_f(z, h, f(z, h))$ is also continuous, so that for μ' satisfying $\mu < \mu' < 1$, where μ is the same as in (3.38), we can shrink U', if necessary, to a neighborhood $U'' \subset U'$ of (z_0, h_0) such that if $(z, h) \in U''$, then

$$\|g_f(z, h, f(z, h))\, s\|_e \leqslant \mu' \|s\|_e \tag{3.44}$$

for every $s \in \mathscr{H}_1^0$ and where $\|\cdot\|_e$ is given by (3.37).

Proposition 3.13. There exists a neighborhood U_1 of (z_0, h_0) and a neighborhood V_1 of f_0 such that if $(z, h) \in U_1$ and if $s \in V_1$ with $s(0) = 1$, then $g^n(z, h, s)$ converges to $f(z, h)$.

Proof. Recall that V is a neighborhood of f_0 such that $g^n(z_0, h_0, f)$ converges to f_0 for every f in V with $f(0) = 1$ and U'' is a neighborhood of (z_0, h_0) such that if (z, h) is in U'', then $f(z, h)$ is in V and (3.44) holds. Let $V' = \{f \in V: f(0) = 1\}$ and let A be the mapping from $U'' \times V'$ to $\mathbf{R}^+$ given by

$$A(z, h, s) = \frac{\|g(z, h, s) - g(z, h, f(z, h)) - g_f(z, h, f(z, h))(s - f(z, h))\|_e}{\|s - f(z, h)\|_e} \tag{3.45}$$

when $s \neq f(z, h)$. If we define $A(z, h, s)$ to be 0 when $s = f(z, h)$, then the proof of Proposition 3.1 [in particular, the estimate (3.11)] immediately gives the continuity of $A(z, h, s)$ in a neighborhood of (z_0, h_0, f_0).

Since A is continuous at (z_0, h_0, f_0), for any $\varepsilon' > 0$, there exists a $\delta > 0$ such that if $(z, h, s) \in B_\delta((z_0, h_0, f_0))$ [the open ball of radius δ around (z_0, h_0, f_0) in $D_\alpha \times S_\alpha \times \mathscr{H}_1$] with $s(0) = 1$, then

$$|A(z, h, f) - A(z_0, h_0, f_0)| < \varepsilon' \tag{3.46}$$

But $A(z_0, h_0, f_0) = 0$, so that (3.46) is equivalent to

$$A(z, h, s) < \varepsilon' \tag{3.47}$$

If we choose $\varepsilon' < 1 - \mu'$, then we have, as in (3.43),

$$\| g^n(z, h, s) - f(z, h) \|_e \leqslant v'^n \, \|s - f(z, h)\|_e \tag{3.48}$$

where $v' = \varepsilon' + \mu' < 1$. Recalling that

$$\|(z, h, f)\|_{\mathbf{C} \times S_\alpha \times \mathscr{H}_1} = \max(\|(z, h)\|_{\mathbf{C} \times S_\alpha}, \|f\|_{\mathscr{H}_1})$$

we have that if $(z, h) \in U_1 = B_\delta((z_0, h_0))$ and if $s \in V_1 = B_\delta(f_0)$ with $s(0) = 1$, then $g^n(z, h, s)$ converges to $f(z, h)$. ∎

Since $g^n(z_0, h_0, 1)$ converges to f_0, there exists an N such that for every $n \geqslant N$, $g^n(z_0, h_0, 1)$ is in V_1. Now $g(z, h, 1)$ is continuous in (z, h) and since $g^n(z, h, 1)$ is just composition n times, it, too, is continuous in (z, h). Thus, there exists a δ_N such that $(z, h) \in B_{\delta_N}((z_0, h_0)) \subset U_1$ implies that $g^N(z, h, 1) \in V_1$. So, for $(z, h) \in B_{\delta_N}((z_0, h_0))$, we have $g^n(z, h, 1)$ converging to $f(z, h)$.

Thus, for $(z, h) \in B_{\delta_N}(z_0, h_0)$ we have that the integrand in (3.3) converges to the analytic function

$$\beta(\varphi^2; z, h)[f(z, h)(TB(z, h) \, f(z, h))](\varphi^2)$$

[analytic in the sense of Definition 3.12 as a mapping from $B_{\delta_N}((z_0, h_0))$ to $\mathscr{H}_1$]. Since $\beta(\varphi^2; z, h)$ decays exponentially and

$$[g^l(z, h, 1)(TB(z, h) \, g^{l-1}(z, h, 1))](\varphi^2)$$

is uniformly bounded in $B_{\delta_N}((z_0, h_0))$, we can apply the dominated convergence theorem to conclude that $G_l(z, h)$ converges to the analytic function $G(z, h)$ given by

$$G(z, h) = \frac{i}{\pi} \int \beta(\varphi^2; z, h)[f(z, h)(TB(z, h) \, f(z, h))](\varphi^2) \, d^2\varphi \tag{3.49}$$

Again the analyticity is in the sense of Definition 3.12 as a mapping from $B_{\delta_N}((z_0, h_0))$ to $\mathbf{C}$. Note that if we fix h, then $G(\cdot, h)$ is an analytic function, in the usual sense, from $B_{\delta_N}(z_0)$ to $\mathbf{C}$.

4. ANALYTICITY OF THE DENSITY OF STATES IN A FINITE STRIP

Theorem 4.1. Let h_0 be the characteristic function of the Cauchy distribution with parameter λ. For any α and α' satisfying $0<\alpha'<\alpha<\lambda$ and any bounded interval I, there exists a neighborhood U around h_0 in the space S_α such that the density of states $\rho(E)$ is analytic in $\{E \in \mathbf{C}: \operatorname{Re} E \in I, |\operatorname{Im} E| < \alpha'\}$.

Proof. In Section 3 it was shown that for every z_0 with $\operatorname{Im} z_0 > -\alpha$ there exists a $\delta>0$ such that in the neighborhood $B_\delta((z_0, h_0))$ around (z_0, h_0), $G_l(z, h)$ converges to $G(z, h)$ [given by Eq. (3.49)] for all $(z, h) \in B_\delta((z_0, h_0))$. Let I be a bounded interval and let $R_{\alpha'} = \{z \in \mathbf{C}: \operatorname{Re} z \in I, |\operatorname{Im} z| < \alpha'\}$. Then we can find a finite number of points z_i, $i = 1,..., m$, in $R_{\alpha'}$ and corresponding positive real numbers $\delta(z_i)$ such that:

(i) $B_{\delta(z_i)}((z_i, h_0))$ is a neighborhood around (z_i, h_0) such that $G_l(z, h)$ converges to $G(z, h)$ for all $(z, h) \in B_{\delta(z_i)}((z_i, h_0))$.

(ii) $\bigcup_{i=1}^m B_{\delta(z_i)}(z_i)$ covers $R_{\alpha'}$.

Take $\delta_{\min} = \min \delta(z_i)$ and let $U = B_{\delta_{\min}}(h_0)$. Then for any $h \in U$ and $z \in R_{\alpha'}$, $G_l(z, h)$ converges to $G(z, h)$ and by the remarks at the end of Section 3, we also have that $G(\cdot, h)$ is analytic in $R_{\alpha'}$ for all $h \in U$. ∎

5. ANALYTICITY OF THE DENSITY OF STATES FOR HIGH ENERGY

Following Constantinescu *et al.*,[15] we derive a random walk expansion for the matrix elements of the resolvent. For $\operatorname{Im} z \neq 0$ we get

$$\langle x| R(z) |y\rangle = \sum_{\omega: x \to y} (-2)^{-n} \prod_{j \in \mathbf{B}} D_j(z)^{-n_j(\omega)} \tag{5.1}$$

where the sum is taken over all random walks starting at x and ending at y, $n_j(\omega)$ is the number of times the walk visits site j, n is the length of the walk and is given by

$$n = \left[\sum_{j \in \mathbf{B}} n_j(\omega)\right] - 1$$

and finally

$$D_j(z) = -z + V(j)$$

The expansion (5.1) will converge absolutely if

$$|\mathrm{Im}\, z| > \sqrt{k} \tag{5.2}$$

Now integrate both sides of (5.1) with respect to $d\bar{\mu} = \prod_{j \in \mathbf{B}} d\mu$:

$$\int \langle x|\, R(z)\, |y\rangle\, d\bar{\mu}(v) = \sum_{\omega: x \to y} (-2)^{-n} \prod_{j \in \mathbf{B}} \int d\mu(v)(-z+v)^{-n_j(\omega)} \tag{5.3}$$

and recall that

$$G(z) = \int \langle x|\, R(z)\, |x\rangle\, d\bar{\mu}(v)$$

We can now prove Proposition 1.2.

Proof of Proposition 1.2. We have from (1.6)

$$G_l(x, y; z) = \langle x|\, (H_l - z)^{-1}\, |y\rangle = \sum_{\substack{\omega: x \to y \\ \omega \text{ stays in } \Lambda_l}} (-2)^{-n} \prod_{j \in \mathbf{B}} [-z + V(j)]^{-n_j(\omega)}$$

where the sum is taken over all walks from x to y which stay in Λ_l. Then

$$|G(x, y; z) - G_l(x, y; z)| = \sum_{\substack{\omega: x \to y \\ \omega \text{ leaves } \Lambda_l}} (-2)^{-n} \prod_{j \in \mathbf{B}} [-z + V(j)]^{-n_j(\omega)}$$

and if $|\mathrm{Im}\, z| > \sqrt{k}$, then the right-hand side of the above equation is just the tail end of an absolutely convergent series.

Now let $A = \{z: \mathrm{Im}\, z > 0\}$. Since

$$|G_l(x, y; E + i\varepsilon)| \leqslant \frac{1}{\varepsilon} \qquad \text{for every} \quad x, y \in \Lambda_l, \quad \varepsilon > 0$$

we have that $G_l(x, y; z)$ is uniformly bounded for $\mathrm{Im}\, z \geqslant \varepsilon'$ for every $\varepsilon' > 0$. Since H_l is self-adjoint, $G_l(x, y; z)$ is analytic on A. Finally, since $G_l(x, y; z) \to G(x, y; z)$ for $\mathrm{Im}\, z > \sqrt{k}$, we can apply Vitali's theorem to conclude that $G_l(x, y; z) \to G(x, y; z)$ on A. ∎

We propose to analytically continue each term on the right-hand side of (5.3) in z beyond the domain specified by (5.2) and across the real axis. We will prove absolute convergence of the analytically continued expansion and this will give an analytic continuation of the left-hand side of (5.3) in z. Since the density of states $\rho(E)$ is the discontinuity of $(1/2\pi i)\, G(z)$ along the real axis, the continued expansion will allow us to analyze $\rho(E)$ as well.

Fix the connectivity k and for $\varepsilon > 0$ let

$$\delta = \tfrac{1}{2}(k+1) + 2\varepsilon \tag{5.4}$$

and

$$r = \tfrac{1}{2}(k+1) + \varepsilon \tag{5.5}$$

Our conditions on h will be stated in terms of the difference $h(t) - h_0(t)$. We will require $h - h_0$ to be in the space S_δ.

For the probability distribution μ with such a characteristic function h we make the following observations (see, e.g., Billingsley[16]):

(i) μ has a density w, i.e., $d\mu(v) = w(v)\, dv$.

(ii) w is analytic in the region $\{v \in \mathbf{C}\colon |\mathrm{Im}\, v| < \delta\} \backslash \{\pm i\lambda\}$.

(iii) $p(E, h) \equiv \sup_{|v-E|=r} |w(h, v)| \to 0$ as $E \to \pm\infty$.

We now establish the convergence of (5.3). Let $z = E + i\eta$, E real, and consider

$$I_n(z) = \int d\mu(v)(-z+v)^{-n}$$

Define

$$\Gamma_1 = \{v\colon |\mathrm{Im}\, v| = 0, \mathrm{Re}\, v \leqslant E - r\}$$

and for $\eta < 0$

$$\Gamma_2^- = \{v\colon |v - E| = r, 0 < \arg(v - E) < \pi\}$$

while for $\eta > 0$

$$\Gamma_2^+ = \{v\colon |v - E| = r, \ -\pi < \arg(v - E) < 0\}$$

Finally

$$\Gamma_3 = \{v\colon |\mathrm{Im}\, v| = 0, \ E + r \leqslant \mathrm{Re}\, v\}$$

and

$$\Gamma^\pm = \Gamma_1 \cup \Gamma_2^\pm \cup \Gamma_3$$

For $\eta < 0$ and $|E| > r$

$$I_n(z) = \int d\mu(v)(-z+v)^{-n} = \int_{\Gamma^-} d\mu(v)(-z+v)^{-n}$$

We now analytically the last integral in the above equation to $\eta < \varepsilon/2$. Then, in this domain, we have

$$\left|\int_{\Gamma_1 \cup \Gamma_3} d\mu(v)(-z+v)^{-n}\right| \leqslant \int_{\Gamma_1 \cup \Gamma_3} d\mu(v)\, r^{-n} \leqslant r^{-n}$$

and

$$\left|\int_{\Gamma_2^-} d\mu(v)(-z+v)^{-n}\right| \leqslant \pi r \left[\frac{1}{2}(k+1) + \frac{\varepsilon}{2}\right]^{-n} \sup_{v \in \Gamma_2^-} |w(v)|$$

So

$$\left|\int_{\Gamma^-} d\mu(v)(-z+v)^{-n}\right| \leqslant \left[\frac{1}{2}(k+1) + \frac{\varepsilon}{2}\right]^{-n} [1 + \pi r p(E, h)]$$

It now follows that given $0 < \varepsilon' < \varepsilon/2$, there exists some finite $E_{\varepsilon'}(h)$ such that for $|E| > E_{\varepsilon'}(h)$ we have

$$|I_n(z)| \leqslant [\tfrac{1}{2}(k+1) + \varepsilon']^{-n} \tag{5.6}$$

Inserting (5.6) into (5.3), we obtain

$$\begin{aligned}
&\left|\int d\bar{\mu}(v)\, \langle x|\, R(z)\, |y\rangle\right| \\
&\quad \leqslant \sum_{\omega: x \to y} 2^{-n} \prod_{j \in \mathbf{B}} |I_{n_j(\omega)}(z)| \\
&\quad \leqslant \sum_{\omega: x \to y} 2^{-n} \prod_{j \in \mathbf{B}} \left[\frac{1}{2}(k+1) + \varepsilon'\right]^{-n_j(\omega)} \\
&\quad = \sum_{\omega: x \to y} 2^{-n} \left[\frac{1}{2}(k+1) + \varepsilon'\right]^{-(n+1)} \\
&\quad = \left[\frac{1}{2}(k+1) + \varepsilon'\right]^{-1} \sum_{\omega: x \to y} [(k+1) + 2\varepsilon']^{-n} \\
&\quad = \left[\frac{1}{2}(k+1) + \varepsilon'\right]^{-1} \sum_{n = |x-y|}^{\infty} \sum_{\substack{\omega: x \to y \\ |\omega| = n}} [(k+1) + 2\varepsilon']^{-n}
\end{aligned}$$

where, as in Section 1, $|x-y|$ is the length of the shortest path from x to y and $|\omega|$ is the length of the walk ω. Now the number of distinct walks of length n starting at a given point is equal to $(k+1)^n$. So

$$\left|\int d\bar{\mu}(v) \langle x|\, R(z)\, |y\rangle\right| \leqslant \left[\frac{1}{2}(k+1)+\varepsilon'\right]^{-1} \sum_{n=|x-y|}^{\infty} \left(1+\frac{2\varepsilon'}{k+1}\right)^{-n}$$

Thus, we have convergence of the analytically continued expansion of the left-hand side of (5.3) for $\eta<0$ and $|E|>E_{\varepsilon'}(h)$. A similar result holds for $\eta>0$. Now the density of states is given by

$$\rho(E)=\frac{1}{2\pi i}\lim_{\eta\downarrow 0}\int d\bar{\mu}(v)[\langle x|\, R(E+i\eta)\, |x\rangle - \langle x|\, R(E-i\eta)\, |x\rangle]$$

$$=\frac{1}{2\pi i}\lim_{\eta\downarrow 0}\left[\int_{\Gamma^+} d\bar{\mu}(v) \langle x|\, R(E+i\eta)\, |x\rangle - \int_{\Gamma^-} d\bar{\mu}(v) \langle x|\, R(E-i\eta)\, |x\rangle\right]$$

For $|E|>E_{\varepsilon'}(h)$ both $\int_{\Gamma^+}$ and $\int_{\Gamma_-}$ converge and are analytic in the strip $|\mathrm{Im}\, E|<\varepsilon'$. We can now easily prove the following result.

Theorem 5.1. Fix k and $\varepsilon>0$ and let δ be given by (5.4). For any ε' in the interval $(0, \varepsilon/2)$ and any $C<+\infty$, there exists some $E(\varepsilon', C)>0$ such that if h satisfies $\|h-h_0\|_{S_\delta}<C$, then $\rho(E)$ is analytic in $\{E\in \mathbf{C}\colon |\mathrm{Re}\, E|>E(\varepsilon', C), |\mathrm{Im}\, E|<\varepsilon'\}$.

Proof. In view of what has been done, it suffices to show that $p(E, h)$ goes to 0 unformly as $E\to \pm\infty$ for all h in $\|h-h_0\|_{S_\delta}<C$. But

$$\begin{aligned}
|w(h, v)| &= \left|\frac{1}{2\pi}\int e^{itv}h(t)\, dt\right| \\
&\leqslant \frac{\lambda}{\pi}\left|\frac{1}{\lambda^2+v^2}\right| + \left|\frac{1}{2\pi}\int e^{itv}(h(t)-h_0(t))\, dt\right| \\
&= \frac{\lambda}{\pi}\left|\frac{1}{\lambda^2+v^2}\right| + \left|\frac{1}{2\pi v}\int e^{itv}(h(t)-h_0(t))'\, dt\right| \\
&\leqslant \frac{\lambda}{\pi}\left|\frac{1}{\lambda^2+v^2}\right| + \left|\frac{C'}{v}\right|
\end{aligned}$$

which immediately gives the uniform convergence of $p(E, h)$ to 0 as $E\to \pm\infty$. ∎

Note that the width of the strip can be made arbitrarily large, but the wider the strip, the stronger the restriction placed on $h-h_0$.

6. ANALYTICITY OF THE DENSITY OF STATES IN A STRIP

We now combine the results of Sections 4 and 5 to prove Theorem 1.1.

Proof of Theorem 1.1. Choose $\delta > \frac{1}{2}(k+1)$, and then take $\varepsilon = \frac{1}{2}[\delta - \frac{1}{2}(k+1)]$. From Theorem 5.1 we know that for any ε' in the interval $(0, \varepsilon/2)$ and $C < +\infty$, there exists an $E(\varepsilon', C) < +\infty$ such that if h is in $\|h - h_0\|_{S_\delta} < C$, then $\rho(E)$ is analytic in $\{E \in \mathbf{C}: |\mathrm{Re}\, E| > E(\varepsilon', C), |\mathrm{Im}\, E| < \varepsilon'\}$.

Now choose $\alpha < \min(\delta, \lambda)$ and $\alpha' \in (0, \alpha)$. Let I be the interval $[-E(\varepsilon', C), E(\varepsilon', C)]$. Then we know from Theorem 4.1 that there exists a neighborhood $B_r(h_0)$, with $r \leqslant C$, around h_0 in the space S_α such that if $h \in B_r(h_0)$, then $\rho(E)$ is analytic in the region $\{E \in \mathbf{C}: \mathrm{Re}\, E \in I, |\mathrm{Im}\, E| < \alpha'\}$. So if $\|h - h_0\|_{S_\delta} < r$, then, since $\delta > \alpha$, it follows that $\|h - h_0\|_{S_\alpha} < r$ and so we have the analyticity of $\rho(E)$ in the strip $|\mathrm{Im}\, E| < \varepsilon''$, where $\varepsilon'' = \min(\varepsilon', \alpha')$. ▌

APPENDIX. THE ERGODIC THEOREM ON THE BETHE LATTICE

Define the level number of a point x, $l(x)$, to be the distance from the origin 0 to x, i.e., $l(x) = |x|$. The points on the Bethe lattice will be labeled as follows:

(i) Label the points whose level number is 1 by (a_1), $a_1 = 1, 2,..., k+1$.

(ii) Label the points whose level number is 2 and whose path to the origin must pass through (a_1) by (a_1, a_2), $a_2 = 1,..., k$.

(iii) Continue the process so that any point x whose level number is l is given by the l-tuple $x = (a_1, a_2,..., a_l)$, where $a_1 = 1,..., k+1$, $a_i = 1,..., k$ for $i > 1$, and whose path to the origin must pass through $(a_1,..., a_{l-1})$, $(a_1,..., a_{l-2})$, etc. (see Fig. 2).

Once the points in $\mathbf{B}$ are labeled as above, let τ_1 be the transformation given by

$$\tau_1 0 = (1)$$

$$\tau_1(a_1,..., a_l) = (1, a_1,..., a_l) \qquad \text{if} \quad 1 \leqslant a_1 \leqslant k$$

$$\tau_1(k+1, a_2,..., a_l) = (a_2 + 1, a_3,..., a_l)$$

The transformation τ_1 essentially "lifts up" the lattice and places the origin at (1) and then orients it by placing (1) at (1, 1), (2) at (1, 2), etc. (see Fig. 3).

Now define τ_2 to be the transformation given by

$$\tau_2(a_1,..., \alpha_l) = ((a_1 + 1) \bmod (k+1), (a_2 + 1) \bmod k,..., (a_l + 1) \bmod k)$$

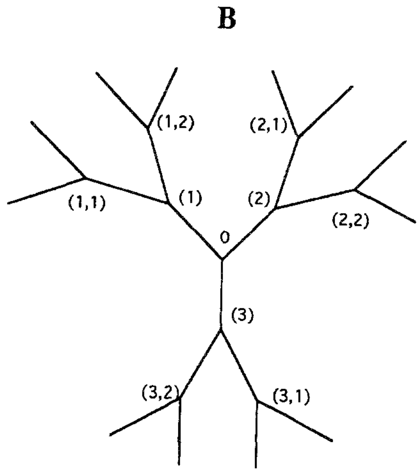

Fig. 2. The labeled Bethe lattice ($k=2$).

The transformation τ_2 can be viewed as a simultaneous rotation about each vertex (see Fig. 3). Note that any point x whose level number is d_1 can be written uniquely as $\tau_2^{d_2}\tau_1^{d_1}0$ with $0 \leqslant d_2 \leqslant (k+1)\,k^{d_1-1}$.

As in ref. 1, define $\bar{f}(\omega) = f(H(\omega)(0,0))$, where f is a continuous function with compact support. Now define the operators T_i by

$$T_i\omega(x) = \omega(\tau_i x)$$

for $i=1, 2$. Using T_1 and T_2, we have, for $x \in \Lambda_l$,

$$f(H(\omega))(x,x) = \bar{f}(T_2^{d_2}T_1^{d_1}\omega) \tag{A.1}$$

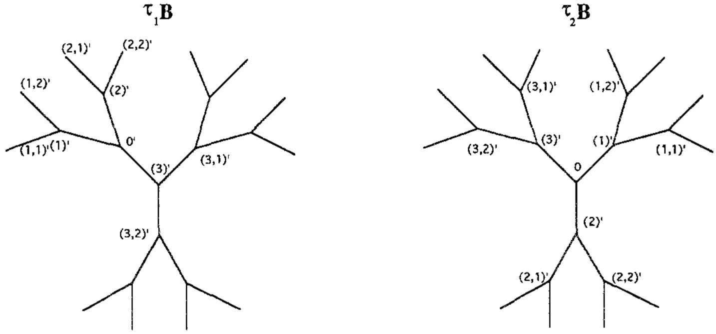

Fig. 3. The transformations τ_1 and τ_2 ($k=2$).

The right-hand side of (1.2) then becomes

$$\frac{1}{|\Lambda_l|}\operatorname{tr}(f(H)\,\chi_l)=\frac{1}{|\Lambda_l|}\sum_{x\in\Lambda_l} f(H(\omega))(x,x)$$

$$=\frac{1}{|\Lambda_l|}\sum_{\substack{0\leqslant d_1\leqslant l\\ 0\leqslant d_2\leqslant (k+1)k^{d_1-1}}} \tilde{f}(T_2^{d_2}T_1^{d_1}\omega) \tag{A.2}$$

We now apply a multiparameter pointwise ergodic theorem due to Zygmund (see ref. 17) to obtain

$$\lim_{l\to\infty}\frac{1}{|\Lambda_l|}\sum_{\substack{0\leqslant d_1\leqslant l\\ 0\leqslant d_2\leqslant (k+1)k^{d_1-1}}} \tilde{f}(T_2^{d_2}T_1^{d_1}\omega)=\mathbf{E}(\mathbf{E}(\tilde{f}(\omega)\,|\mathscr{F}_1)|\,\mathscr{F}_2)\quad\text{a.e.} \tag{A.3}$$

where $\mathscr{F}_1$ and $\mathscr{F}_2$ are the σ-algebras of the invariant sets of T_1 and T_2, respectively.

Noting that T_1 is ergodic, it follows that

$$\mathbf{E}(\mathbf{E}(\tilde{f}(\omega)\,|\mathscr{F}_1)\,|\mathscr{F}_2)=\mathbf{E}(\tilde{f}(\omega))=\mathbf{E}(f(H(\omega))(0,0)) \tag{A.4}$$

giving the a.e. convergence of $\int f\,dN_l$ to $\int f\,dN$. The vague convergence of dN_l to dN is given in, for example, ref. 1.

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